

The Trade-offs of Domain Adaptation for Neural Language Models

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Abstract

This work connects language model adaptation with concepts of machine learning theory. We consider a training setup with a large out-of-domain set and a small in-domain set. We derive how the benefit of training a model on either set depends on the size of the sets and the distance between their underlying distribution. We also present how adaptation technique based on data selection such as importance sampling, intelligent data selection and influence functions, can be presented in a common framework which highlights their similarity and also their subtle differences.

1 Introduction

Neural language models (LMs) are central to modern language processing (Bommasani et al., 2021), either as standalone models (e.g. speech recognition, machine translation, summarization) or to provide initialization to other models (e.g. sentiment analysis, question answering, parsing). Models trained over large generic training sets – over a billion sentence (Kaplan et al., 2020; Roziowski and Kozłowski, 2021) – have shown to be effective, especially when adapted to the targeted application domain. Adaptation takes into account that the training set for most targeted domains are orders of magnitude smaller than generic sets. Neural LM adaptation is commonly performed via fine tuning (Devlin et al., 2018; Liu et al., 2019; Raffel et al., 2019; Radford et al., 2019), data selection (van der Wees et al., 2017) or their combination (Wang et al., 2018; Aharoni and Goldberg, 2020; Gururangan et al., 2020).

It has been observed that the out-of-domain pre-training set size is a factor of success (Raf-

fel et al., 2019; Devlin et al., 2018). Also it has been mentioned that adaptation is more effective on domains which are well represented in the the pre-training data (Radford et al., 2019). In this paper, we connect language model adaptation with concepts of machine learning theory. We consider a training setup with a large out-of-domain set and a small in-domain set. As a first contribution, we derive how the benefit of training a model on either set depends on the size of the sets and the distance between their underlying distribution. We also expose how fine-tuning can be viewed as regularization methods that can achieve a better trade-off than training only on either set.

The research on data selection for language model adaptation originates mainly from intelligent selection (Moore and Lewis, 2010; Axelrod et al., 2011). This method favors training data deemed more likely by an in-domain model than by an out-of-domain model. This method is intuitive and is effective for neural LM adaptation (van der Wees et al., 2017; Wang et al., 2018). However, this method connection with statistical estimation is unclear, which makes studying its impact on generalization error difficult. Another family of selection methods stem from influence functions (Koh and Liang, 2017; Wang et al., 2021) which estimates if the model updates from out-of-domain training examples are aligned with the in-domain updates. This method is more principled and its impact on the generalization error easier to study. In this work, as a second contribution, we show how intelligent selection and importance sampling methods are linked in the case of neural LMs. In particular, we show that they both can be derived from importance sampling (Owen, 2013), a classical, well-studied statistical estimation technique.

* Work performed while interning at Google.

This paper first introduces language models and studies the trade-offs of in-domain and out-of-domain training. It then presents LM fine-tuning as a regularization method before drawing the link between data selection techniques.

2 Neural Language Modeling

Language modeling refers to the generative modeling of natural language (Manning and Schütze, 1999). Commonly, natural language is represented as a sequence of symbols, tokens, from a finite vocabulary. For instance, language can be represented as a sequence of characters, a sequence of words or alternative units. A neural language model (LM) decomposes the estimates the log probability of a text $y = (y_1, \dots, y_n)$, as

$$\log P(y; \theta) = \sum_{i=1}^n \log P(y_i | y_1^{i-1}; \theta)$$

where P_θ maps a parameter vector θ along with a sequence of past tokens y_1^{i-1} onto a probability distribution over the vocabulary. Different types of neural architectures have been used for neural language modeling. Most architectures used for language models re-use intermediate computations from the previous steps for the next steps when estimating probabilities for successive tokens in the same sequence. Popular architectures include recurrent neural networks (Mikolov et al., 2010; Sundermeyer et al., 2012), convolutional networks (Dauphin et al., 2017) and transformer networks (Vaswani et al., 2017; Radford et al., 2019).

The parameter vector $\theta \in \Theta$ of a neural language model is identified by maximizing the log likelihood over a training set D sampled from the true distribution \mathcal{D} using variants of stochastic gradient descent. The log likelihood of a held-out set, sampled from the same distribution, is commonly used to evaluate model quality. It is common to report the exponentiated negative average log likelihood per token, called perplexity.

Conditional LMs model the conditional distribution of a text y given a conditioning input x .

$$\log P(y|x; \theta) = \sum_{i=1}^n \log P(y_i | y_1^{i-1}, x; \theta)$$

This type of model is used for translation where (x, y) pairs are sentences in the source and target language (Koehn, 2009; Bahdanau et al., 2015) or summarization where (x, y) pairs are corresponding articles and summaries (See et al., 2017).

For both conditional and regular LMs, the size of the training data is an important factor to achieve a low held-out perplexity. This is an obstacle for domains where training data is available only in limited quantity. This issue has led to various solutions in the model adaptation literature. These methods leverage large amount of generic training data D along with a small amount of target domain training data T from the domain of interest. *Fine tuning* is a popular domain adaptation method which trains a neural language model in two phases, first maximizing the likelihood of the generic set D (pre-training) before optimizing the likelihood of the target domain set T (fine-tuning). As an alternative to fine-tuning, some methods consider leveraging the small target-domain training set to identify and emphasize similar data in the larger generic training set. These *emphasis* methods can be used individually or in conjunction with fine-tuning.

Emphasis methods include importance sampling, contrastive data selection and influence functions. This paper shows that these methods – although proposed in different context – can be presented in a unified way which allows casting light on their subtle differences. Moreover, we show that using these methods in conjunction with fine-tuning require considering multiple types of over-fitting risks.

3 In-Domain Training

This section decomposes the generalization loss of language models when the training and test data are sampled from the same distribution. This decomposition relies on classical concepts from learning theory and will help us study the domain adaptation setup in the next sections.

To train a language model over a training set D sampled from a distribution \mathcal{D} , one typically minimizes the negative log-likelihood of

D , also referred to as the *cross-entropy* loss i.e.

$$\begin{aligned}\mathcal{L}(\theta; D) &= -\frac{1}{|D|} \sum_{y \in D} \log P(y|\theta) \\ &= \mathbb{E}_{y \sim D} [-\log P(y|\theta)].\end{aligned}$$

This *empirical risk* is an expectation over the finite set D , which acts as a proxy for the expectation over the true, unavailable distribution $P(y; \mathcal{D})$,

$$\begin{aligned}\mathcal{L}(\theta; \mathcal{D}) &= -\sum_{y \in \Omega} \log P(y|\theta)P(y|\mathcal{D}) \\ &= \mathbb{E}_{y \sim \mathcal{D}} [-\log P(y|\theta)].\end{aligned}$$

The true expected loss is bounded by the entropy of the distribution $P(\cdot|\mathcal{D})$, i.e.

$$\mathcal{L}(\theta; \mathcal{D}) \geq \mathcal{L}_H(\mathcal{D}) = H(P(\cdot|\mathcal{D}))$$

since $H(P(\cdot|\mathcal{D})) = \min_q \mathbb{E}_{y \sim \mathcal{D}} [-\log q(y)]$. The gap between the best likelihood from a neural network with the chosen parameterization and the entropy is called the approximation error

$$\mathcal{L}_{\text{app}}(\mathcal{D}, \Theta) = \min_{\theta \in \Theta} \mathcal{L}(\theta; \mathcal{D}) - H(P(\cdot|\mathcal{D})).$$

This gap accounts for the fact that $P(\cdot|\mathcal{D})$ generally cannot be represented by a parameterized function from the chosen family spanned by Θ . In addition to the approximation error, one should consider the estimation error to account for the fact that one relies on the empirical risk from the finite set D ,

$$\mathcal{L}_{\text{est}}(\mathcal{D}, \Theta, D) = \mathcal{L}(\theta_D; \mathcal{D}) - \min_{\theta} \mathcal{L}(\theta; \mathcal{D})$$

with $\theta_D = \arg \min_{\theta \in \Theta} \mathcal{L}(\theta; \mathcal{D})$. Therefore, the loss of θ_D over \mathcal{D} decomposes as (Bottou and Bousquet, 2007)

$$\boxed{\mathcal{L}(\theta_D; \mathcal{D}) = \mathcal{L}_H(\mathcal{D}) + \mathcal{L}_{\text{app}}(\mathcal{D}, \Theta) + \mathcal{L}_{\text{est}}(\mathcal{D}, \Theta, D)} \quad (1)$$

where the three terms accounts for the intrinsic uncertainty of \mathcal{D} , the chosen neural architecture and the finite training set D respectively.

The approximation error $\mathcal{L}_{\text{app}}(\mathcal{D}, \Theta)$ depends on the selected model family Θ . It can be reduced by selecting a more expressive family, i.e. a neural architecture with

more capacity, a larger Θ , e.g. architectures with more, wider layers. The estimation error $\mathcal{L}_{\text{est}}(\mathcal{D}, \Theta, D)$ depends both on the selected model family Θ and the size of the training data D . Increasing model capacity will result in a higher estimation error for the same training set size, but training over a larger training set will decrease estimation error. Therefore, for a given training set size, capacity needs to be chosen to identify a good trade-off between the two error types.

Two important properties of neural networks need to be kept in mind when examining this trade-offs. The *universal approximation* property (Lecun, 1987; Funahashi, 1989) means that for any approximation error ϵ and any distribution \mathcal{D} , there exists a capacity setting $C(\epsilon, \mathcal{D})$ at which a neural network $\theta \in C(\epsilon, \mathcal{D})$ whose error is below ϵ , i.e.

$$\forall \epsilon > 0, \exists C \text{ s.t. } \mathcal{L}_{\text{app}}(\mathcal{D}, C) \leq \epsilon.$$

In layman terms, the *universal approximation* property means that for sufficiently large capacity settings, the approximation error can become arbitrary low. The *statistical consistency* property means that for any $\epsilon, \epsilon' > 0$, there exist a training set size $N(\epsilon, \mathcal{D})$ such that sampling a training set of size $N(\epsilon, \epsilon', \mathcal{D})$ from \mathcal{D} will result in an estimation error less than ϵ' with probability $1 - \epsilon$,

$$\forall \epsilon, \epsilon' > 0, \exists N \text{ s.t.}$$

$$P(D \sim \mathcal{D}^N : \mathcal{L}_{\text{est}}(\mathcal{D}, \Theta, D) < \epsilon') = 1 - \epsilon$$

In layman terms, the *statistical consistency* property means that for sufficiently large training sets, the probability to get an estimation error below any positive value can become arbitrary close to 1.

Universal approximation and consistency implies that, in the asymptotic case (i.e. as the size of D tends to infinity), the last two terms in Eq 1 can be arbitrary close to zero with the appropriate model capacity (with high probability). In that case, the likelihood $\mathcal{L}(\theta_D; \mathcal{D})$ amount to the intrinsic entropy of \mathcal{D} with the appropriate model capacity.

4 Out-of-Domain Training

This section considers a setup where one needs a specialized language model in a domain \mathcal{T}

and two training sets are available: a small training set T sampled from \mathcal{T} and a large training set D sampled from \mathcal{D} , a generic domain different from the specialized domain.

In that context, the simplest options are either to train a model over T or D alone. Training only on the small set T results in the generalization loss

$$\mathcal{L}(\theta_T; \mathcal{T}) = \mathcal{L}_H(\mathcal{T}) + \mathcal{L}_{\text{app}}(\mathcal{T}, \Theta) + \mathcal{L}_{\text{est}}(\mathcal{T}, \Theta, T)$$

with $\theta_T = \arg \min_{\theta \in \Theta} \mathcal{L}(\theta; T)$ as in the previous section. Training on the larger set D results in

$$\mathcal{L}(\theta_D; \mathcal{T}) = \mathcal{L}_H(\mathcal{T}) + \mathcal{L}_{\text{app}}(\mathcal{T}, \Theta) + \mathcal{L}_{\text{est}}(\mathcal{T}, \Theta, D).$$

Two factors are important to compare these two options: the size of the specialized set T relative to the size of the generic set D and the similarity between \mathcal{T} and \mathcal{D} distributions.

When the \mathcal{T} and \mathcal{D} distributions are identical, D and T are sampled from the same distribution and training a model on the larger training set D is advantageous. For a constant capacity, this option will get a lower estimation error. When varying capacity, one might identify a setting with an even better trade-off in the compound loss with the larger training set D .

When the distributions \mathcal{T} and \mathcal{D} differ and the size of D is fixed, the size of T determines which option to prefer. Statistical consistency means that $\mathcal{L}_{\text{est}}(\mathcal{T}, \Theta, T)$ will converge to zero in probability as the size of T grows. This means that when the size of T is greater than $N(\epsilon, \mathcal{L}_{\text{est}}(\mathcal{T}, \Theta, D), \mathcal{D})$, the probability that training on T results in a better generalization loss than training on D is above $1 - \epsilon$.

Theorem 1 When the distributions \mathcal{T} and \mathcal{D} differ, the Kullback–Leibler (KL) divergence between the two distributions is considered. We show that the generalization of the loss of θ_D over T is upper bounded

$$\boxed{\forall \epsilon > 0, \exists N \text{ s.t. } \forall D \sim \mathcal{D}^n, \mathcal{L}(\theta_D; \mathcal{T}) \leq H(\mathcal{T}) + KL(\mathcal{T}, \mathcal{D}) + \epsilon} \quad (2)$$

with probability $1 - \epsilon$. This bound justifies the intuition that if given the choice between two generic domains \mathcal{D} and \mathcal{D}' , training over the one with the lowest KL divergence to \mathcal{T}

will result in a better asymptotic behaviour. The proof of this bound is presented in Appendix A.

5 Fine-Tuning

Fine-tuning for domain adaptation trains a model on a small in-domain set initializing optimization from the parameters of a model trained on a large out-of-domain set. Formally, fine-tuning minimizes $\mathcal{L}(\theta; T)$ the loss over T for a few steps, starting the optimization from $\theta_D = \arg \min_{\theta \in \Theta} \mathcal{L}(\theta; D)$. This strategy implicitly targets a trade-off between the empirical losses over T and D . This trade-off is controlled by the number of fine tuning steps n_{ft} . Few steps means that the identified parameters θ_{ft} achieve a low loss over D , while many steps expresses that the parameters achieve a low loss over T . This strategy leverage the regularization effect of early stopping (Caruana et al., 2001), i.e. the solution found by gradient descent is guaranteed to be in an Euclidean ball centered around the initialization whose radius grows with the number of steps (Grangier and Bengio, 2008), i.e.

$$\|\theta_{\text{ft}} - \theta_D\|_2 \leq \lambda n_{\text{ft}} g_{\text{max}}$$

where λ refers to the (maximum) learning rate and g_{max} to an upper bound on the update norm, which is commonly set to 1 for transformers trained with gradient clipping (Vaswani et al., 2018). The small distance between θ_{ft} and θ_D guarantees that the loss $\mathcal{L}(\theta_{\text{ft}}; D)$ is close to the optimum $\mathcal{L}(\theta_D; D)$ when $\theta \rightarrow \mathcal{L}(\theta; D)$ is a smooth function, e.g. a Lipschitz function.

For the basic fine-tuning setup, several variants have been introduced. Some approaches (Devlin et al., 2018; Liu et al., 2019; Raffel et al., 2019) consider leaving some parameters un-tuned or frozen which is the extreme case of regularization for these weights, penalizing any deviation from initialization. Other approaches consider introducing novel (unregularized) weights for fine tuning, often referred as *adapter* layers (Houlsby et al., 2019; Stickland et al., 2019; Pfeiffer et al., 2020). Other form of regularization such as dropout have also been considered in conjunction with fine tuning (Miceli Barone et al., 2017).

The selection of the regularization strength in fine-tuning is efficient since it iteratively visits an optimization path from the most regularized model (θ_D trained only on D , Sec. 4) to the unregularized θ_T (Sec. 3). This is more efficient compared to explicit regularization methods, including multitask learning (Caruana, 1998; Collobert and Weston, 2008; Pilault et al., 2021), i.e. optimizing $\mathcal{L}_{\text{multi}}(\theta; T, D, \alpha) = \mathcal{L}(\theta; T) + \alpha\mathcal{L}(\theta; D)$.

6 Data Selection

The generalization error of a language model trained over a generic training set D sampled from \mathcal{D} depends on the test data distribution. Its generalization over another specialized domain \mathcal{T} is tied to the KL divergence between \mathcal{T} and \mathcal{D} , see Eq. (4). When this KL divergence is large but out-of-domain data is abundant, data selection methods propose to select a subset $D^{\mathcal{T}} \subset D$. Ideally, the training loss over such a subset $\mathcal{L}(\theta, D^{\mathcal{T}})$ would be a better proxy for the generalization loss over \mathcal{T} , $\mathcal{L}(\theta, \mathcal{T})$, than the initial training loss over D , $\mathcal{L}(\theta, D)$.

Data selection is attractive since it replaces the training set with another set closer to the test domain. At the same time, this training set is smaller, which increases the impact of estimation errors. Additionally, data selection is imperfect since the target domain distribution \mathcal{T} is only known through a small target training set T .

This section connects importance sampling, contrastive data selection and influence filtering, alternative selection techniques introduced independently.

6.1 Importance Sampling

Although selecting data from the generic training set is more common (Moore and Lewis, 2010; Wang et al., 2018) than re-weighting and re-sampling, we first examine importance sampling, a re-weighting and re-sampling technique. This section will guide our understanding of other selection methods later.

Importance sampling is a generic statistical technique (Owen, 2013). In our case, it can be used to estimate the expectation of the cross-entropy loss over \mathcal{T} while having access to sam-

ples from \mathcal{D} . It relies on the identity

$$\begin{aligned} \mathcal{L}(\theta; \mathcal{T}) &= \mathbb{E}_{y \sim \mathcal{T}}[-\log P(y|\theta)] \\ &= -\sum_{y \in \Omega} \log P(y|\theta) P(y|\mathcal{T}) \\ &= -\sum_{y \in \Omega} \log P(y|\theta) \frac{P(y|\mathcal{T})}{P(y|\mathcal{D})} P(y|\mathcal{D}) \\ &= \mathbb{E}_{y \sim \mathcal{D}}[-w(y; \mathcal{T}, \mathcal{D}) \log P(y|\theta)] \end{aligned}$$

where $w(y; \mathcal{T}, \mathcal{D}) = \frac{P(y|\mathcal{T})}{P(y|\mathcal{D})}$, assuming full support on \mathcal{D} , i.e. $\forall y \in \Omega, P(y|\mathcal{D}) > 0$. In practice, one has not access to \mathcal{T} and \mathcal{D} but to finite samples T and D . With importance sampling, we can consider two alternative estimators of $\mathcal{L}(\theta; \mathcal{T})$, either the empirical risk over T ,

$$\mathcal{L}(\theta; T) = -\frac{1}{|T|} \sum_{y \in T} \log P(y|\theta)$$

or the mean of the importance weighted cross entropy over D , i.e.

$$\mathcal{L}_{\text{imp}}(\theta; D, T, \hat{w}) = -\frac{1}{|D|} \sum_{y \in D} \hat{w}(y; \mathcal{T}, \mathcal{D}) \log P(y|\theta)$$

where \hat{w} estimates of the weights w from the training sets D and T . The trade-off between these two estimators of the generalization loss over T depends on multiple factors: the relative size of T and D , the imbalance of the weights w and the quality of their estimate \hat{w} .

Importance sampling is interesting when the generalization error $\mathcal{L}(\theta_{\text{imp}(D,T)}; \mathcal{T})$ of the model

$$\theta_{\text{imp}(D,T)} = \arg \min_{\theta} \mathcal{L}_{\text{imp}}(\theta; D, T, \hat{w})$$

is less than the generalization error of θ_T selected by minimizing $\mathcal{L}(\theta; T)$, i.e. classical empirical risk minimization. This error decomposes as,

$$\begin{aligned} \mathcal{L}(\theta_{\text{imp}(D,T)}; \mathcal{T}) &= \\ &= \mathcal{L}_H(\mathcal{T}) + \mathcal{L}_{\text{app}}(\mathcal{T}, \Theta) + \mathcal{L}_{\text{est}}^{\text{imp}}(\mathcal{T}, \Theta, D, T). \end{aligned}$$

We further decompose the estimation error in two terms,

$$\begin{aligned} \mathcal{L}_{\text{est}}^{\text{imp}}(\mathcal{T}, \Theta, D, T) &= \\ &= \mathcal{L}_{\text{est/w}}(\mathcal{T}, \mathcal{D}, \Theta, D) + \mathcal{L}_{\text{est/\hat{w}}}(\mathcal{T}, \Theta, D, T) \end{aligned}$$

where $\mathcal{L}_{\text{est}/w}(\mathcal{T}, \mathcal{D}, \Theta, D)$ refers to the estimation error resulting from the finite size of D , assuming access to the true importance weights, and $\mathcal{L}_{\text{est}/\hat{w}}(\mathcal{T}, \Theta, D, T)$ isolate the residual error resulting from the estimation of w . We have

$$\mathcal{L}_{\text{est}/w}(\mathcal{T}, \mathcal{D}, \Theta, D) = \mathcal{L}(\theta_{\text{imp}(D, \mathcal{D})}; \mathcal{D}) - \min_{\theta} \mathcal{L}(\theta; \mathcal{T})$$

and

$$\mathcal{L}_{\text{est}/\hat{w}}(\mathcal{T}, \Theta, D, T) = \mathcal{L}(\theta_{\text{imp}(D, \mathcal{T})}; \mathcal{D}) - \mathcal{L}(\theta_{\text{imp}(D, T)}; \mathcal{D})$$

with $\theta_{\text{imp}(D, \mathcal{D})} = \arg \min_{\theta} \mathcal{L}_{\text{imp}}(\theta; D, T, \hat{w})$

The first term depends on the size of D and the imbalance of weights. For instance, if the weights are mostly concentrated over a small subset of D , this estimation error will be high. If this subset is smaller than T , estimation errors from $\mathcal{L}_{\text{imp}}(\theta; D, T, \hat{w})$ will be higher than from $\mathcal{L}(\theta; T)$. The notion of *effective sample size* has been defined to quantify this effect (Kish, 1965). It is defined by examining the variance of the weighted sum of n independent random variables Z_i with mean μ_Z and variance σ_Z^2 , $S_w = \frac{\sum_i w_i Z_i}{\sum_i w_i}$. This variance is

$$\sigma_{S_w}^2 = \frac{\sum_i w_i^2}{(\sum_i w_i)^2} \sigma_Z^2$$

which can be compared to $\sigma_S^2 = \frac{1}{n} \sigma_Z^2$ in the unweighted case. This means that the weighted sum variance matches the variance of an unweighted case with

$$n_e = \frac{(\sum_i w_i)^2}{\sum_i w_i^2}.$$

Assuming that losses over \mathcal{D} and \mathcal{T} have comparable means and variances, the expected loss estimate with importance weighting over D has lower variance than the mean over T only when,

$$n_e = \frac{(\bar{w})^2}{\overline{w^2}} |D| \gg |T|$$

where $\bar{w} = \frac{1}{|D|} \sum_{y \in D} w(y)$ and $\overline{w^2} = \frac{1}{|D|} \sum_{y \in D} w^2(y)$ are the sample mean and variance of the weights over D . This means that the first term in the estimation error is $\mathcal{L}_{\text{est}/w}(\mathcal{T}, \Theta, D, T)$ advantageous compared to

the estimation error from classical empirical risk minimization over T when T is small. In this case, importance sampling seems attractive for small specialized sets.

Unfortunately, the second estimation error term $\mathcal{L}_{\text{est}/\hat{w}}(\mathcal{T}, \Theta, D, T)$ gets larger as T gets smaller since estimating the importance weights $w(y; \mathcal{T}, \mathcal{D}) = \frac{P(y|\mathcal{T})}{P(y|\mathcal{D})}$ from data is challenging when T is small. One can remark that language modeling is actually the very problem of estimating the probabilities from this ratio $P(y|\mathcal{T}), P(y|\mathcal{D})$ given finite samples from \mathcal{T}, \mathcal{D} . Discriminative classifiers are also relevant to estimate this ratio since

$$w(y; \mathcal{T}, \mathcal{D}) \propto \frac{P(\mathcal{T}|y)}{P(\mathcal{D}|y)}.$$

In fact the multiplying constant (prior ratio) does not matter since multiplying the weighted loss by a positive constant has no impact on optimization.

When importance weights are estimated with a language model, one can estimate $P(\cdot|\mathcal{T})$ by fine tuning on T a model pre-trained on D . The number of tuning steps n_{ft} gives controls on $\|\theta_{\text{ft}} - \theta_D\|$. When $n_{\text{ft}} = 0$, $\hat{w} = 1$ and the importance sampling loss corresponds to $\mathcal{L}(\theta, D)$. As n_{ft} grows, the estimate $P(y|\theta^{\text{ft}})$ could overfit and assigns most of the probability mass to a small neighborhood around samples in T . The weights \hat{w} will in turn be concentrated in this small neighborhood, making the minimizer of the importance sampling loss close to the minimizer of the empirical loss over T . Therefore, fine-tuning a language model for estimating the importance weights allow to progressively transition between the in-domain and the out-of-domain empirical loss minimizers seen in Section 4. In the next sections, we refer to the estimated importance sampling weights as

$$w_{D, T}^{\text{imp}}(y) = \hat{w}(y; T, D).$$

Importance sampling has been used for model training for various application: either to improve training speed (Johnson and Guestrin, 2018; Katharopoulos and Fleuret, 2018) or to adapt to a changing training distribution (Mahmood et al., 2014; Metelli et al., 2018). Importance sampling has rarely been used to modify the training distribution of language models (Foster et al., 2010; Fernandez

and Downey, 2018) and intelligent selection methods are more common.

6.2 Intelligent Selection

Intelligent selection (Moore and Lewis, 2010; Axelrod et al., 2011) and contrastive data selection, its extension to neural networks (van der Wees et al., 2017; Wang et al., 2018), have been introduced in the language modeling literature. We show that these methods are closely related to importance sampling, even if their original papers does not mention this link.

Intelligent selection selects training samples from an out-of-domain dataset according to the log-odd between an in-domain language model and an out-of-domain language model. Typically, a binary decision is taken per sentence by comparing the average log-odd to a threshold τ ,

$$\mathcal{L}^{\text{IntSel}}(\theta, D, T) = - \sum_{y \in D} b_{D,T}^{\text{IntSel}}(y) \log P(y|\theta)$$

where $b_{D,T}^{\text{IntSel}}(y)$ is defined as $\mathbb{I}\{\log P(y|\theta_T) - \log P(y|\theta_D) > \tau\}$. Compared to importance sampling, the weights are binarized, i.e.

$$b_{D,T}^{\text{IntSel}}(y) = \mathbb{I}\{\log w_{D,T}^{\text{imp}}(y) > \tau\}.$$

The binarization decision was certainly driven by convenience as most n-gram language models training packages did not support weighted likelihood optimization when intelligent selection was introduced. Binarization also has the advantage of down-weighting extreme positive weight values from large $\log P(y|\theta_T)$ due to over-fitting on the small set T .

More recently, intelligent selection has been extended to neural models (van der Wees et al., 2017; Wang et al., 2018). Contrastive data selection (Wang et al., 2018) suggests to fine tune the in-domain model $\log P(y|\theta_T)$ from $\log P(y|\theta_D)$ and also observes that selection scores can efficiently be estimated from a model with a much smaller capacity than the final trained model. Dynamic selection (van der Wees et al., 2017) proposes to increase the selection threshold τ_t as training progresses, gradually transitioning from generic to in-domain training. This gradual

adaptation of neural network is related to curriculum learning (Bengio et al., 2009) which studies the ordering of examples and tasks during model training.

Intelligent selection methods have been applied both for unconditional models (language modeling) and conditional models (machine translation). In the conditional case, intelligent selection computes

$$b_{D,T}^{\text{IntSel}}(x, y) = \mathbb{I}\{\log w_{D,T}^{\text{IntSel}}(x, y) > \tau\}$$

$$\text{with } w_{D,T}^{\text{IntSel}}(x, y) = \frac{P(y|x, \theta_T)}{P(y|x, \theta_D)}.$$

This ratio of conditional probabilities is different from the ratio of joint probabilities stemming from importance sampling, i.e.

$$\mathcal{L}_{\text{imp}}(\theta; D, T, \hat{w}) = - \frac{1}{|D|} \sum_{y \in D} \frac{P(x, y|\mathcal{T})}{P(x, y|\mathcal{D})} \log P(y|x, \theta).$$

The two ratio matches when $P(x|\mathcal{T}) = P(x|\mathcal{D})$ since

$$w_{D,T}^{\text{imp}}(x, y) = \frac{P(x, y|\mathcal{T})}{P(x, y|\mathcal{D})} = \frac{P(x|\mathcal{T})}{P(x|\mathcal{D})} w_{D,T}^{\text{IntSel}}(x, y).$$

The formulation of intelligent selection therefore neglects the domain mismatch from the input distribution in the conditional case. This formulation aligns with the denoising goal (Wang et al., 2018) which assumes that D contains label noise, i.e. mistranslation in that case.

6.3 Influence Functions

As mentioned above, importance sampling and intelligent selection weights can be estimated by contrasting the log probabilities from a base model with those from a fine-tuned model. This use of fine-tuning connects intelligent selection to influence function and gradient alignment techniques. Influence functions (Koh and Liang, 2017; Pruthi et al., 2020) have been used as a diagnostic tool to identify the training instances which support or contradict with a given test label. This task is related to the selection of training data when

the objective is to find instances in a generic training set D whose training updates increase the likelihood of a set T from a different domain.

The influence of a training point y on a test point y' is defined as

$$\mathbf{I}(y, y') = -\frac{\partial \ell}{\partial \theta}(y'; \theta)^\top H^{-1} \frac{\partial \ell}{\partial \theta}(y; \theta)$$

where $\ell(y, \theta)$ refers to the loss at y for a model with parameters θ and H_θ refers to the Hessian of the model loss at θ . This quantity can be derived by considering the impact of reducing the weight of point y during training on the test loss at y' . If we increase the weight of a training example by ϵ ,

$$\theta_{D, \epsilon} = \min_{\theta} \frac{1}{|D|} \sum_{z \in D} \ell(z; \theta) + \epsilon \ell(y; \theta)$$

From (Cook and Weisberg, 1982), we derive

$$\left. \frac{\partial \theta_{D, \epsilon}}{\partial \epsilon} \right|_{\epsilon=0} = -H^{-1} \left. \frac{\partial \ell}{\partial \theta}(y; \theta) \right|_{\theta=\theta_D}.$$

Composing with the test loss on x', y' , we get

$$\left. \frac{\partial \ell(y'; \theta_{D, \epsilon})}{\partial \epsilon} \right|_{\epsilon=0} = - \left. \frac{\partial \ell(y'; \theta)}{\partial \theta} \right|_{\theta=\theta_D}^\top H^{-1} \left. \frac{\partial \ell(y; \theta)}{\partial \theta} \right|_{\theta=\theta_D}$$

which matches the expression of influence introduced above.

We now connect influence with the precedent sections on importance sampling and contrastive data selection. We consider a language models with weights θ_D , trained on the generic training set D . Its first order Taylor expansion at θ_D is

$$\begin{aligned} \log P(y|\theta_D + \Delta\theta) &= \\ \log P(y|\theta_D) + \Delta\theta^\top g(y; \theta_D) + O(\|\Delta\theta\|^2) \end{aligned} \quad (3)$$

where $g(y; \theta_D) = \frac{\partial}{\partial \theta} \log P(y|\theta)|_{\theta=\theta_D}$. If the model pre-trained on D is fine-tuned on T by performing a single step of gradient descent with learning rate λ , we get

$$\begin{aligned} \theta_T &= \theta_D - \lambda \left. \frac{\partial}{\partial \theta} \mathcal{L}(T; \theta) \right|_{\theta=\theta_D} \\ &= \theta_D + \lambda \mathbb{E}_{y \sim T} [g(y; \theta_D)]. \end{aligned}$$

In that case, the log-odd of the two models therefore has the following Taylor expansion,

$$\begin{aligned} \log P(y|\theta_T) - \log P(y|\theta_D) &= \\ \lambda \mathbb{E}_{y' \sim T} [g(y'; \theta_D)^\top g(y; \theta_D)] + O(\|\theta_D - \theta_T\|^2). \end{aligned}$$

If we assume that the model hessian is the identity, $H_\theta = \mathbb{1}$, we therefore have

$$\begin{aligned} \log P(y|\theta_T) - \log P(y|\theta_D) &= \\ -\lambda \mathbb{E}_{y' \sim T} [\mathbf{I}(y, y')] + O(\|\theta_D - \theta_T\|^2). \end{aligned}$$

The hessian assumption might be dropped when the model is fine-tuned with a Newton-style update (Boyd and Vandenberghe, 2014). The above relation means that the negative mean influence of a point $y \in D$ over the set T also corresponds to the log of the estimated importance weights introduced in Section 6.1, i.e.

$$\begin{aligned} \log w_{D, T}^{\text{imp}}(y) &= \\ -\lambda \mathbb{E}_{y' \sim T} [\mathbf{I}(y, y')] + O(\|\theta_D - \theta_T\|^2). \end{aligned}$$

Of course, this relation holds only in the case where a single gradient step is performed for fine-tuning. This relation allows estimating the reduction in test loss (here over T) when removing training samples from D with positive influence which is also the goal of intelligent data selection. This strategy has been applied to label noise filtering (Koh and Liang, 2017), class rebalancing (Ren et al., 2018) and domain adaptation (Wang et al., 2021).

7 Conclusions

This work focuses on domain adaptation for neural language modeling. It compares the generalization properties of a model trained over a large out-of-domain corpus as opposed to a model trained over a small in-domain corpus. It shows how fine-tuning, the most common approach for neural LM adaptation can achieve better trade-offs than either solutions. We then focus on data selection techniques, i.e. techniques to emphasize in-domain data in an out-of-domain training set and show that common techniques can all be derived from importance sampling.

This theoretical paper rises interesting empirical questions we aim to answer in future work: e.g. the binarization of importance sampling weights in intelligent selection is a simple variance reduction technique and more sophisticated alternative might be beneficial empirically. Influence functions suggest that examples with importance sampling weights lower

than one have only a negative effect on the in-domain likelihood but this might not be confirmed empirically...

In addition to empirical questions, our analysis should be extended to cover more application settings. In particular, we did not consider adaptation scenarios where novel adapter layers are introduced during fine-tuning. Neither did we consider cases when one might be interested in a generalization error different from the training loss: for instance machine translation training often relies on negative log likelihood but other metrics are used for evaluation.

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A Proof of Theorem 1

When the distributions \mathcal{T} and \mathcal{D} differ, the Kullback–Leibler (KL) divergence between the two distributions is considered. We show that the generalization of the loss of θ_D over \mathbb{T} is upper bounded

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall D \sim \mathcal{D}^n, \\ \mathcal{L}(\theta_D; \mathcal{T}) \leq H(\mathcal{T}) + KL(\mathcal{T}, \mathcal{D}) + \epsilon \quad (4)$$

with probability $1 - \epsilon$. This bound justifies intuition that if given the choice between two generic domain \mathcal{D} and \mathcal{D}' , training over the one with the lowest KL divergence to \mathcal{T} will result in better asymptotic behaviour.

Proof. We consider the asymptotic case for the size of D . For any $\epsilon > 0$, the universal approximation property allows us to consider a model capacity large enough such that

$$\mathcal{L}_{\text{app}}(\mathcal{D}, \Theta) < \frac{\epsilon}{2}$$

Using consistency, we can also consider a training set D large enough such that

$$\mathcal{L}_{\text{est}}(\mathcal{D}, \Theta, D) < \frac{\epsilon}{2}$$

with probability $1 - \epsilon$. With the same probability,

$$\mathcal{L}(\theta_D; \mathcal{D}) < \mathcal{L}_H(\mathcal{D}) + \epsilon$$

which can be rewritten as a bound on the Kullback-Leibler divergence,

$$KL(P(\cdot|\mathcal{D}), P(\cdot|\theta_D)) = \mathcal{L}(\theta_D; \mathcal{D}) - \mathcal{L}_H(\mathcal{D}) < \epsilon.$$

This bound can help connecting the generalization loss of θ_D over \mathbb{T} with the Kullback-Leibler divergence of \mathbb{T} and \mathcal{D} ,

$$\begin{aligned} & \mathcal{L}(\theta_D; \mathcal{T}) \\ &= \sum_{y \in \Omega} P(y|\mathcal{T}) \log P(y|\theta_D) \\ &= \sum_{y \in \Omega} P(y|\mathcal{T}) \log(P(y|\mathcal{D}) + P(y|\theta_D) - P(y|\mathcal{D})) \\ &\leq \sum_{y \in \Omega} P(y|\mathcal{T}) \log(P(y|\mathcal{D}) + |P(y|\mathcal{D}) - P(y|\theta_D)|) \\ &\leq \sum_{y \in \Omega} P(y|\mathcal{T}) \log(P(y|\mathcal{D}) + 2\epsilon^2) \quad (5) \\ &\leq \sum_{y \in \Omega} P(y|\mathcal{T}) \log(P(y|\mathcal{D})) + \log(1 + 2m\epsilon^2) \\ &\leq H(\mathcal{T}) + KL(\mathcal{T}, \mathcal{D}) + \log(1 + 2m\epsilon^2) \end{aligned}$$

where $m = 1/\min_y P(y|\mathcal{D})$ assumes that $P(\cdot|\mathcal{D})$ has full support, and (5) relies on Pinsker's inequality, i.e. $\max_y |P(y) - Q(y)| < 2KL(Q, Y)^2$. \square